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Classical Lorentz-Invariant Theory of Systems with Self-Action: Lagrangian Formulation

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A Lorentz-invariant theory of singular Lagrangian particle systems with self-action, treated as a dependence of the Lagrangian upon acceleration, is developed. The Lagrangian equations and their exact special solutions are found. The first integrals of 4-momentum and angular momentum are calculated. Particles possessing weak self-action are treated as classical analogs of particles with half-integral spin.

I. The purpose of this paper is to develop a Lorentz-invariant theory of Lagrangian systems containing, besides coordinates and velocities, their derivatives of higher orders. The presence of such derivatives describes a self-action, being equivalent to the existence in the system of an exterior field whose characteristics are determined, in their turn, by derivatives of the second order. The first integrals of the 4-momentum and angular momentum are found for these systems. Systems in which the self-action is weak are studied in more detail; similar investigations have been carried out by Mathisson (1937) and Papapetrou (1951).

We show here that the admission of self-action leads to qualitatively new physical effects which do not follow from a conventional Lorentz-invariant theory in the pseudo-Euclidean space.

II. In the pseudo-Euclidean space with metric[1]

$$ds^2 = \varepsilon_{\mu)} \quad \delta_{\mu\nu}\, dx^\mu\, dx^\nu \tag{1}$$

where $\varepsilon_{\mu)} = \mathrm{diag}(1, -1, -1, -1)$ is Eisenhart's symbol and $\delta\mu\nu$ is Kronecker's

[1] $\mu, \nu = 0, 1, 2, 3$. Summation over the greek indices is performed according to the Einstein rule.

symbol, we consider a particle described by an action integral of the form

$$S = -\alpha \int_{s_1}^{s_2} \mathfrak{L}(-w_\nu w^\nu)\, ds \tag{2}$$

Here the integration is performed along a world line between two given events, i.e., locations of the system at its initial and final positions at given time moments τ_1 and τ_2; α is a constant characterizing the system. Later on for simplicity we put $\alpha = 1$.

We denote 4-velocity by $u^\mu = dx^\mu/ds$ and 4-acceleration by $w^\mu = du^\mu/ds$; it is obvious that

$$u_\mu u^\mu = 1, \qquad u_\mu w^\mu = 0 \tag{3}$$

The form (2) of the action integral is chosen for two reasons: firstly, it must be an integral of a true scalar; secondly, the relations (3) lead to the fact that the function $\mathfrak{L}$ may depend only on $(-w_\nu w^\nu)$, the sign being chosen for convenience since

$$w_\nu w^\nu < 0 \tag{4}$$

It is assumed also that the integrand does not depend explicitly on coordinates and time.

We do not take into account relations (3) in advance, so when varying S no use of the method of indefinite Lagrangian multipliers has to be made.

According to the principle of least action,

$$\delta S = 0 \tag{5}$$

After variation, taking into account that the operations δ and d/ds do not commute, and representing ds as $ds = \surd(dx_\nu\, dx^\nu)^{1/2}$, we obtain

$$\delta S = -\delta \int \mathfrak{L}\, ds = -\int \delta\mathfrak{L}\, ds - \int \mathfrak{L}\,\delta\, ds \tag{6}$$

$$\delta\mathfrak{L} = \delta w^\mu \cdot (\partial\mathfrak{L}/\partial w^\mu) \tag{7}$$

$$\delta w^\mu = \delta(du^\mu/ds) = d(\delta u^\mu)/ds - (du^\mu/ds)(\delta ds/ds) \tag{8}$$

$$\delta u^\mu = \delta(dx^\mu/ds) = d(\delta x^\mu)/ds - u^\mu(\delta ds/ds) \tag{9}$$

$$\delta ds = \frac{dx_\mu d(\delta x^\mu)}{ds} = u_\mu\, d(\delta x^\mu) \tag{10}$$

Substituting equations (6)–(10) into (5) and taking into account that

$$\delta x^{\mu}\Big|_{s_1, s_2} = \delta u^{\mu}\Big|_{s_1, s_2} = 0 \tag{11}$$

we get

$$\delta S = -\int \delta x_{\mu} \left\{ \frac{d^2}{ds^2} \frac{\partial \mathfrak{L}}{\partial w_{\mu}} - \frac{d}{ds} \left[u^{\mu} \left(\mathfrak{L} + u_{\nu} \frac{d}{ds} \frac{\partial \mathfrak{L}}{\partial w_{\nu}} - w_{\nu} \frac{\partial \mathfrak{L}}{\partial w_{\nu}} \right) \right] \right\} ds = 0 \tag{12}$$

From this and from the arbitrariness of δx_{μ}, the Lagrangian equations

$$\frac{d^2}{ds^2} \frac{\partial \mathfrak{L}}{\partial w_{\mu}} = \frac{d}{ds} \left[u^{\mu} \left(\mathfrak{L} + u_{\nu} \frac{d}{ds} \frac{\partial \mathfrak{L}}{\partial w_{\nu}} - w_{\nu} \frac{\partial \mathfrak{L}}{\partial w_{\nu}} \right) \right] \tag{13}$$

follow.

The equations (13) lead to the first integral, the 4-vector of the system's momentum:

$$p^{\mu} = u^{\mu} \left(\mathfrak{L} + u_{\nu} \frac{d}{ds} \frac{\partial \mathfrak{L}}{\partial w_{\nu}} - w_{\nu} \frac{\partial \mathfrak{L}}{\partial w_{\nu}} \right) - \frac{d}{ds} \frac{\partial \mathfrak{L}}{\partial w_{\mu}} \tag{14}$$

Let us return now to the variation δS.

When the system is moving along a true trajectory, the equations (13) are satisfied identically, and δS has a form

$$\delta S = -\int \frac{d}{ds} \left[p_{\mu} \delta x^{\mu} + \frac{\partial \mathfrak{L}}{\partial w_{\mu}} \delta u_{\mu} \right] ds \tag{15}$$

where p^{μ} is determined by the expression (14).

If the system is translated infinitesimally, then

$$\delta x^{\mu} = \delta \tilde{x}^{\mu} + a^{\mu}, \qquad \delta u^{\mu} = 0 \tag{16}$$

where a^{μ} is an infinitesimal constant vector, then, as the system is closed, we come to the conservation law for the 4-momentum p^{μ}. If the system undergoes an infinitesimal rotation, we have

$$\delta x^{\mu} = \delta \Omega^{\mu\nu} x_{\nu}, \qquad \delta u^{\mu} = \delta \Omega^{\mu\nu} u_{\nu} \tag{17}$$

where $\delta \Omega^{\mu\nu} = -\delta \Omega^{\nu\mu}$ is an infinitesimal skew-symmetrical tensor (Landau and Lifshitz, 1967)

$$\delta S = -\int \delta\Omega^{\mu\nu} \frac{d}{ds}\left[p_\mu x_\nu + \frac{\partial \mathfrak{L}}{\partial w^\mu} u_\nu \right] ds \tag{18}$$

from which, taking into consideration that $\delta\Omega^{\mu\nu} = -\delta\Omega^{\nu\mu}$ is arbitrary, the system is closed and[2]

$$\delta\Omega^{\mu\nu}\left(p_{[\mu,} x_{\nu]_+} + \frac{\partial \mathfrak{L}}{\partial w^{[\mu,}} u_{\nu]_+} \right) := 0 \tag{19}$$

we come to the conservation law for the skew-symmetrical 4-tensor of the angular momentum of the system:

$$M^{\mu\nu} = \frac{1}{2}\left(p^{[\nu,} x^{\mu]_-} + \frac{\partial \mathfrak{L}}{\partial w_{[\nu,}} u^{\mu]_-} \right) \tag{20}$$

which differs from the classical expression

$$M^{\mu\nu}_{\text{or}} = \tfrac{1}{2} p^{[\nu,} x^{\mu]_-} \tag{21}$$

by the presence of a spin term

$$M^{\mu\nu}_{\text{s}} = \frac{1}{2} \frac{\partial \mathfrak{L}}{\partial w_{[\nu,}} u^{\mu]_-} \tag{22}$$

III. We rewrite now equations (13) in terms of the Frenet tetrad $\{e^\mu_{(\alpha)}\}$ which is determined by conditions

$$e^\mu_{(0)} = u^\mu, \qquad e^\mu_{(i)} e_{\mu(k)} = g_{(i)(k)} = \operatorname{diag}(-1, -1, -1)$$

$$(i, k = 1, 2, 3) \tag{23}$$

$$\frac{d}{ds} e^\mu_{(0)} = w^\mu = y e^\mu_{(1)}, \qquad \frac{d}{ds} e^\mu_{(1)} = y e^\mu_{(0)} + z e^\mu_{(2)}$$

$$\frac{d}{ds} e^\mu_{(2)} = -z e^\mu_{(1)} + t e^\mu_{(3)}, \qquad \frac{d}{ds} e^\mu_{(3)} = -t e^\mu_{(2)} \tag{23a}$$

where y, z, t are parameters of the tetrad,

$$-w_\nu w^\nu = y^2 \tag{24}$$

$$\frac{\partial \mathfrak{L}}{\partial w_\mu} = -2 w^\mu \frac{\partial \mathfrak{L}}{\partial(-w_\nu w^\nu)} = -2 y e^\mu_{(1)} \frac{\partial \mathfrak{L}}{\partial y^2} \tag{25}$$

[2] Here $p_{[\nu,} x_{\mu]+} = p_\nu x_\mu + p_\mu x_\nu$, $p_{[\nu,} x_{\mu]_-} = p_\nu x_\mu - p_\mu x_\nu$.

and below

$$\frac{\partial}{\partial y^2} := ' \tag{26}$$

Because of equations (22)–(26), the Lagrangian equation (13) takes the form

$$\begin{aligned} &e^{\mu}_{(0)}\left[\frac{d}{ds}(\pounds - 2y^2\pounds') + 2y\frac{d}{ds}(y\pounds')\right] \\ &+ e^{\mu}_{(1)}\left[y(\pounds - 2y^2\pounds' - 2z^2\pounds') + 2\frac{d^2}{ds^2}(y\pounds')\right] \\ &+ e^{\mu}_{(2)}\left[2z\frac{d}{ds}(y\pounds') + \frac{d}{ds}(2\pounds' yz)\right] \\ &+ e^{\mu}_{(3)}[2\pounds' yzt] = 0 \end{aligned} \tag{27}$$

so that

$$\frac{d}{ds}(\pounds - 2y^2\pounds') + 2y\frac{d}{ds}(y\pounds') = 0 \tag{28}$$

$$y(\pounds - 2y^2\pounds' - 2z^2\pounds') + 2\frac{d^2}{ds^2}(y\pounds') = 0 \tag{29}$$

$$2z\frac{d}{ds}(y\pounds') + \frac{d}{ds}(2\pounds' yz) = 0 \tag{30}$$

$$2yzt\pounds' = 0 \tag{31}$$

Since the parameter t (having nothing to do with the time) appears only in (31), the solution of this equation is $t = 0$.

We shall seek a solution of the system of equations (28)–(30) in the form

$$\frac{dy}{ds} = \frac{dz}{ds} = 0 \tag{32}$$

Then equations (28) and (30) are satisfied identically, and equation (31) is transformed from a differential equation to an algebraic one which includes z as an arbitrary constant parameter.

The system of equations (28)–(30) takes the form

$$y(\pounds - 2y^2\pounds' - 2z^2\pounds') = 0 \tag{33}$$

One of the solutions of the equation is

$$y=0 \tag{34}$$

This corresponds to

$$w^{\mu}=\frac{du^{\mu}}{ds}=ye^{\mu}_{(1)}=0, \qquad u^{\mu}=\text{const} \tag{35}$$

and describes a system with action integral in a form

$$S=-\int ds \tag{36}$$

with the 4-momentum

$$p^{\mu}=u^{\mu}, \qquad M^{\mu\nu}=0 \tag{37}$$

If $y \neq 0$, then solutions to the equation

$$\mathfrak{L}-2y^2\mathfrak{L}'-2z^2\mathfrak{L}'=0 \tag{38}$$

depend on the specific form of the function $\mathfrak{L}(y^2)$ which will be considered later.

Let us turn to the expression (14) for the 4-momentum p^{μ}.

Substituting the Frenet tetrad into equation (14) and using equations (32) and (38), we obtain

$$p^{\mu}=e^{\mu}_{(0)}(2\mathfrak{L}'z^2)+e^{\mu}_{(2)}(2\mathfrak{L}'yz) \tag{39}$$

It is easily seen from equation (39) that, if the torsion parameter z vanishes, then $p^{\mu}=0$ for any form of the function $\mathfrak{L}$.

After contraction the equation (39) takes the form

$$p_{\mu}p^{\mu}=4\mathfrak{L}'^2z^2(z^2-y^2) \tag{40}$$

from which it follows that the sign of $p_{\mu}p^{\mu}$ depends on the relation between $|z|$ and $|y|$.

1. Let $|z|<|y|$. In this case the Frenet tetrad can be chosen as

$$\begin{aligned}
e^{\mu}_{(0)}&=\left[\left(h^{\mu}_{(0)}\cosh\lambda+h^{\mu}_{(1)}\sinh\lambda\right)\cosh\theta+h^{\mu}_{(2)}\sinh\theta\right]\cosh\psi+h^{\mu}_{(3)}\sinh\psi\\
e^{\mu}_{(1)}&=\left(h^{\mu}_{(0)}\cosh\lambda+h^{\mu}_{(1)}\sinh\lambda\right)\sinh\theta+h^{\mu}_{(2)}\cosh\theta\\
e^{\mu}_{(2)}&=\left[\left(h^{\mu}_{(0)}\cosh\lambda+h^{\mu}_{(1)}\sinh\lambda\right)\cosh\theta+h^{\mu}_{(2)}\sinh\theta\right]\sinh\psi+h^{\mu}_{(3)}\cosh\psi
\end{aligned} \tag{41}$$

Here ψ, λ are arbitrary constant parameters, $\theta = \theta(s)\{h^{\mu}_{(\alpha)}\}$ is a constant orthonormal tetrad of the pseudo-Euclidian manifold.

The relations (23a) take the form

$$y = \cosh\psi \frac{d\theta}{ds} \tag{42}$$

$$z = -\sinh\psi \frac{d\theta}{ds} \tag{43}$$

$$\cosh\psi = \left(1 - \frac{z^2}{y^2}\right)^{-1/2}, \qquad \sinh\psi = -\frac{z}{y}\left(1 - \frac{z^2}{y^2}\right)^{-1/2} \tag{44}$$

Introducing the absolute time $c\tau = x^{(0)}$ and performing integration, we obtain

$$x^{(0)} = \sinh\theta\left(\cosh^2\psi\cosh\lambda/y\right), \qquad x^{(1)} = \sinh\theta\left(\cosh^2\psi\sinh\lambda/y\right)$$

$$x^{(2)} = \cosh\theta\left(\cosh^2\psi/y\right), \qquad x^{(3)} = \theta\left(\sinh\psi\cosh\psi/y\right) \tag{45}$$

Using equations (41)–(44), the formula (39) for p^{μ} takes a form

$$p^{\mu} = h^{\mu}_{(3)}(2\mathfrak{L}'yz)\left(1 - z^2/y^2\right)^{1/2} \tag{46}$$

i.e., in the laboratory reference frame p^{μ} has only one spatial component.

Let us turn now to the equations (20)–(22) for the 4-tensor of angular momentum, $M^{\mu\nu}$. We take of it the purely spatial part M^{ik}, and form the 3-vector of angular momentum[3],

$$M^{i} = e^{ikl}M_{kl}, \qquad i, k, l = 1,2,3 \tag{47}$$

Here e^{ikl} is the Levi-Cività symbol, completely skew-symmetrical object of the third rank. Using equations (20–(22), we find the expression for M^{i}:

$$M^{i} = e^{ikl}\cdot\tfrac{1}{2}\left(p_{[e,}x_{k]\_} + \frac{\partial\mathfrak{L}}{\partial w}{}_{[e,}u_{k]\_}\right) \tag{48}$$

It can be easily shown by direct calculation using equations (39)–(45) that the orbital part of the moment

$$M^{ik}_{\text{or}} = \tfrac{1}{2}p^{[k,}x^{i]\_} \tag{49}$$

[3]Latin indices are subjected to the usual summation convention.

identically vanishes. Therefore, only the spin part

$$M_s^{ik} = \frac{1}{2} \frac{\partial \mathfrak{L}}{\partial w_{[k,}} u^{i]-} \tag{50}$$

makes a contribution to the 3-vector of the angular momentum. From equation (48) we obtain

$$M^i = h^i_{(3)} \left[y\mathfrak{L}' \sinh\lambda \left(1 - \frac{z^2}{y^2}\right)^{-1/2} \right] \tag{51}$$

2. Let $|y| < |z|$. In this case the Frenet tetrad can be chosen in the form

$$\begin{aligned} e^\mu_{(0)} &= \left[h^\mu_{(0)}\cosh\lambda + h^\mu_{(3)}\sinh\lambda\right]\cosh\psi + \left[h^\mu_{(1)}\cos\theta + h^\mu_{(2)}\sin\theta\right]\sinh\psi \\ e^\mu_{(1)} &= -h^\mu_{(1)}\sin\theta + h^\mu_{(2)}\cos\theta \\ e^\mu_{(2)} &= \left[h^\mu_{(0)}\cosh\lambda + h^\mu_{(3)}\sinh\lambda\right]\sinh\psi + \left[h^\mu_{(1)}\cos\theta + h^\mu_{(2)}\sin\theta\right]\cosh\psi \end{aligned} \tag{52}$$

where ψ, λ are arbitrary parameters, $\theta = \theta(s)$.

Performing similar calculations, we obtain

$$y = \sinh\psi \frac{d\theta}{ds}, \qquad z = -\cosh\psi \frac{d\theta}{ds} \tag{53}$$

$$\cosh\psi = \left(1 - \frac{y^2}{z^2}\right)^{-1/2}, \qquad \sinh\psi = -\frac{y}{z}\left(1 - \frac{y^2}{z^2}\right)^{-1/2} \tag{54}$$

$$\begin{aligned} x^{(0)} &= \theta(\cosh\lambda\cosh\psi\sinh\psi/y), \qquad x^{(1)} = \sin\theta(\sinh^2\psi/y) \\ x^{(2)} &= -\cos\theta(\sinh^2\psi/y), \qquad x^{(3)} = \theta(\sinh\lambda\cosh\psi\sinh\psi/y) \end{aligned} \tag{55}$$

Now the expression (14) for p^μ takes a form

$$p^\mu = \left[h^\mu_{(0)}\cosh\lambda + h^\mu_{(3)}\sinh\lambda\right](2\mathfrak{L}'z^2)\left(1 - \frac{y^2}{z^2}\right)^{1/2} \tag{56}$$

i.e., in this case p^μ contains both temporal and spatial components.

For M^{ik} we obtain again

$$M^{ik}_{\text{or}} = \tfrac{1}{2} p^{[k,} x^{i]-} = 0 \tag{57}$$

and only the spin part

$$M_s^i = h_{(3)}^i \left[-\mathfrak{L}' y^2 z^{-1} \left(1 - \frac{y^2}{z^2}\right)^{-1/2} \right] \tag{58}$$

makes a contribution to M^i.

3. The case $|z| = |y|$ corresponds to $ds = 0$ and is not considered in this paper.

IV. Let us consider specific forms of $\mathfrak{L}$ functions.

We shall restrict ourselves to the case when the self-action in a system is weak and the function $\mathfrak{L}$ can be represented as

$$\mathfrak{L} = 1 + \varepsilon f(y^2) \tag{59}$$

Here $f(y^2)$ is an arbitrary function, and ε is a small parameter, the meaning and value of which are determined by the type of self-action in the system. Obviously, the type of the function $f(y^2)$ will determine the power of the algebraic equation (38). We shall consider the cases when the equation (38) is linear one. The functions

$$\mathfrak{L} = 1 + \varepsilon (y^2)^{1/2} \tag{60}$$

$$\mathfrak{L} = 1 + \varepsilon y^2 \tag{61}$$

meet this requirement.

Substituting the function (60) into equation (38) $\mathfrak{L} - 2y^2\mathfrak{L}' - 2z^2\mathfrak{L}' = 0$, we obtain

$$1 + \varepsilon (y^2)^{1/2} - 2y^2(\varepsilon/2)(y^2)^{-1/2} - 2z^2(y^2)^{-1/2} = 0 \tag{62}$$

or

$$y = \varepsilon z^2 \tag{63}$$

We put $|z| < 1$. This is justified by the necessity of the limiting transition $|z| \to 0$. Then $z^2 < |z|$ and $y = \varepsilon z^2 < |z|$. Substituting equation (63) into the expressions (56) for p^μ and (58) for M^i we obtain

$$p^\mu = \left[h_{(0)}^\mu \cosh\lambda + h_{(3)}^\mu \sinh\lambda \right] \cdot (1 - \varepsilon^2 z^2)^{1/2} \tag{64}$$

$$M^i = h_{(3)}^i \left[-\frac{\varepsilon^2 z}{2} (1 - \varepsilon^2 z^2)^{-1/2} \right] \tag{65}$$

$$p_\mu p^\mu = 1 - \varepsilon^2 z^2 > 0 \tag{66}$$

Making the limiting transition in equations (63)–(66), $\varepsilon, z \to 0$ we come to the case described by equations (34)–(37).

Let us turn now to the function (61).

Substituting (63) into equation (38), we obtain

$$1+\varepsilon y^2-2\varepsilon y^2-2\varepsilon z^2=0 \tag{67}$$

or

$$y^2=\frac{1-2\varepsilon z^2}{\varepsilon}>z^2 \tag{68}$$

We calculate p^μ and M^i from equations (46) and (51) taking into account equation (68):

$$p^\mu=2h^\mu_{(3)}\left(|\varepsilon|z^2\right)^{1/2}\left(1-3\varepsilon z^2\right)^{1/2} \tag{69}$$

$$M^i=h^i_{(3)}\left[|\varepsilon|^{1/2}\sinh\lambda\left(1-2\varepsilon z^2\right)\left(1-3\varepsilon z^2\right)^{-1/2}\right] \tag{70}$$

$$p_\mu p^\mu=-4|\varepsilon|z^2\left(1-3\varepsilon z^2\right)<0 \tag{71}$$

Taking a similar limiting transition in equations (69)–(71), $\varepsilon, z \to 0$ we obtain

$$p^\mu=0, \qquad M^i=0 \tag{72}$$

It is worth mentioning that if only $z \to 0$, then $p^\mu=0$ and

$$M^i=h^i_{(3)}\left(\sinh\lambda|\varepsilon|^{1/2}\right) \tag{73}$$

V. Turning to the discussion of the results of this paper, we conclude that the system described by the Lagrangian

$$\mathfrak{L}=1+\varepsilon\left(y^2\right)^{1/2}$$

may serve as a classical analog of quantum-mechanical free particles with half-integral spin. Here the presence of a spin angular momentum is conditioned by two factors: the purely geometrical one, z (torsion), and the physical one, ε, that characterizes the type of interaction in the system. These same factors lead, independent of signs of parameters, to decreasing the square of the 4-momentum of the system.

The case described by the Lagrangian $\mathfrak{L} = 1 + \varepsilon y^2$ does not have any classical or quantum-mechanical analogs since the usual 3-velocity in the given solution is less than the light velocity, though the squared 4-momentum is negative. This property differs drastically from the so-called "tachyon" solutions.

Systems with Lagrangians $\mathfrak{L}$ of different forms are not considered in this paper; the complication of $\mathfrak{L}$ leads to algebraic equations of higher degrees, and in the cases when the latter have real roots one comes to a model theory of interacting particles.

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